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Information theoretic approach to statistical properties of multivariate Cauchy–Lorentz distributions

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Abstract

The Cauchy–Lorentz (CL) distribution has divergent lowest moments. This necessarily leads to the fact that the information theoretic approach is essential for the study of its statistical properties. Here, correlation measured by the mutual entropy is discussed for the multivariate CL distribution. It is found that correlation obeys a simple scaling law with respect to the dimensionality of the distribution. Then, regarding the CL distribution as a power-law quantum wavepacket, the information entropic uncertainty relation is also discussed both analytically and numerically. It is found that the sum of the position and momentum information entropies tends to the value of the lower bound for large dimensions.

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1. Introduction

The Cauchy–Lorentz (CL) distribution

$$p(x) = \frac{1}{\pi}\frac{1}{1+x^2} \qquad (x \in \boldsymbol{R}) \tag{1}$$

is frequently encountered in physics. It is well known that the spectral distributions of a radiation emitted through the transition of an atom (or a nucleus) with finite life time, a radiation field in a cavity or a random-phase radiation field [1] are of the CL-type. Another example we wish to mention is the momentum distribution of an electron–hole pair trapped in a uniform quantum well [2], which has the form of the bivariate generalization of equation (1). Mathematically, the distribution in equation (1) belongs to the (symmetric) Lévy stable class, $L_\alpha(x) = (2\pi)^{-1}\int \mathrm{d}k\ \exp(-\mathrm{i}kx - a\,|k|^\alpha)$, where α is the Lévy index satisfying $0 < \alpha < 2$ and a is a positive constant. For large values of $|x|$, it behaves as $L_\alpha(x) \sim |x|^{-1-\alpha}$.

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The CL distribution corresponds to the 'ballistic limit', $\alpha \to 1$. A feature of these power-law distributions is that their lowest moments are divergent. This fact makes it nontrivial to quantify their statistical properties.

In this paper, we study the statistical properties of the multivariate CL distribution based on the information theoretic approach. To the best of our knowledge, this issue has not been addressed yet in the literature. Specifically, we discuss correlation in this distribution measured by the mutual entropy. It is shown that the mutual entropy exhibits a simple scaling behaviour with respect to the dimensionality of the distribution. Then, we also discuss the CL distribution in the quantum mechanical context and calculate the information entropic uncertainty both analytically and numerically. We shall see that the sum of the position and momentum information entropies tends to the value of the lower bound for large dimensions. Throughout this paper, all relevant random variables are assumed to be made dimensionless.

2. Multivariate Cauchy–Lorentz distributions

Let us define the multivariate CL distribution [3] as follows:

$$p(\boldsymbol{x}) = p(x_1, x_2, \ldots, x_N) = \frac{\Gamma\left(\frac{N+1}{2}\right)}{\pi^{(N+1)/2}} \frac{1}{\left(1 + x_1^2 + x_2^2 + \cdots + x_N^2\right)^{(N+1)/2}} \qquad (x_i \in \boldsymbol{R}) \tag{2}$$

where $\Gamma(z)$ is the gamma function. Clearly, equation (1) is the case when $N = 1$. An interesting property of this distribution is that its marginal distribution is again the CL distribution with the reduced number of variables. In particular, the marginal distribution of a single variable is found to be given by

$$\begin{aligned} p^{(i)}(x_i) &= \int \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}\hat{x}_i \cdots \mathrm{d}x_N\, p\ (x_1, x_2, \ldots, x_N) \\ &= \frac{1}{\pi} \frac{1}{1 + x_i^2} \qquad (i = 1, 2, \ldots, N) \end{aligned} \tag{3}$$

where the integral over x_i is excluded. This is the single-variable CL distribution in equation (1).

3. Correlation measured by mutual entropy

The distribution in equation (1) does not admit a factorized form, and therefore there exists correlation between the random variables. However, since all moments are mathematically ill defined, it is not possible to quantify correlation using covariance and variance. To overcome this difficulty, we examine the information theoretic approach.

In information theory, it is known that the Kullback–Leibler relative entropy [4,5] defined by

$$K_N\left[p_1 \| p_2\right] = \int \mathrm{d}^N \boldsymbol{x}\, p_1(\boldsymbol{x}) \ln \frac{p_1(\boldsymbol{x})}{p_2(\boldsymbol{x})} \tag{4}$$

measures the difference of p_1 from the reference distribution, p_2. It satisfies

$$K_N[p_1 \| p_2] \geqslant 0 \tag{5}$$

for any distributions, p_1 and p_2. The equality holds if and only if $p_1 = p_2$. Therefore, the degree of correlation in p_1 can be measured by the Kullback–Leibler entropy of p_1 with respect to the product of its marginal distributions, $p_2 = p^{(1)} p^{(2)} \cdots p^{(N)}$:

$$K_N\left[p \| \, p^{(1)} p^{(2)} \cdots p^{(N)}\right] = \int \mathrm{d}^N \boldsymbol{x}\, p(\boldsymbol{x}) \ln \frac{p(\boldsymbol{x})}{p^{(1)}(x_1) p^{(2)}(x_2) \cdots p^{(N)}(x_N)}. \tag{6}$$

This quantity is referred to as the mutual entropy.

In our problem, $p(\boldsymbol{x})$ and $p^{(i)}(x_i)$ $(i = 1, 2, \ldots, N)$ are given in equations (2) and (3), respectively. Substituting them into equation (6) and performing the integration explicitly, we obtain the following result:

$$K_N\left[p \,\middle\|\, p^{(1)} p^{(2)} \cdots p^{(N)}\right] = \ln \Gamma\left(\frac{N+1}{2}\right) + \frac{N-1}{2} \ln 4\pi - \frac{N+1}{2}\left[\gamma + \psi\left(\frac{N+1}{2}\right)\right] \quad (7)$$

where $\psi(z)$ stands for the digamma function [6] defined by $\psi(z) = \mathrm{d} \ln \Gamma(z) / \mathrm{d} z$ and $\gamma = -\psi(1) = 0.5772\ldots$ is Euler's constant [6]. It is of interest to examine how this mutual entropy behaves asymptotically for large N. Using Stirling's formula [6], we see

$$\ln \Gamma\left(\frac{N+1}{2}\right) \cong \frac{N}{2} \ln \frac{N}{2} - \frac{N}{2} \quad (8)$$

whereas, for the digamma function, we have [6]

$$\psi\left(\frac{N+1}{2}\right) \cong \ln \frac{N}{2}. \quad (9)$$

Therefore, we find for large N

$$K_N\left[p \,\middle\|\, p^{(1)} p^{(2)} \cdots p^{(N)}\right] \cong \frac{N}{2}(\ln 4\pi - \gamma - 1). \quad (10)$$

Thus, we conclude that the degree of correlation in the multivariate CL distribution scales linearly with respect to the dimensionality of the distribution, asymptotically.

4. Information entropic uncertainty in multivariate Cauchy–Lorentz wavepackets

Next, we wish to study information entropic uncertainty in the context of quantum mechanics. Here, the distribution in equation (2) is regarded as the modulus squared of the wavepacket

$$\phi(\boldsymbol{x}) \equiv \langle \boldsymbol{x} | \phi \rangle = \frac{\sqrt{\Gamma\left(\frac{N+1}{2}\right)}}{\pi^{(N+1)/4}} \frac{1}{\left(1 + x_1^2 + x_2^2 + \cdots + x_N^2\right)^{(N+1)/4}}. \quad (11)$$

This is a higher-dimensional generalization of the so-called Cauchy wavepacket discussed in [7, 8]. Its momentum representation (i.e. the Fourier transformation) is found to be

$$\begin{aligned} \tilde{\phi}(\boldsymbol{p}) \equiv \langle \boldsymbol{p} | \phi \rangle &= \frac{1}{(2\pi)^{N/2}} \int \mathrm{d}^N \boldsymbol{x}\, \mathrm{e}^{\mathrm{i} \boldsymbol{p} \cdot \boldsymbol{x}} \phi(\boldsymbol{x}) \\ &= \frac{\sqrt{\Gamma\left(\frac{N+1}{2}\right)}}{2^{(N-3)/4} \pi^{(N+1)/4} \Gamma\left(\frac{N+1}{4}\right)} \frac{K_{(N-1)/4}(|\boldsymbol{p}|)}{|\boldsymbol{p}|^{(N-1)/4}} \end{aligned} \quad (12)$$

where $K_\nu(z)$ $(\mathrm{Re}\, \nu > -1/2)$ is the modified Bessel function [6] and $\hbar$ is set equal to unity for the sake of simplicity.

Now, for measurements of the position $\boldsymbol{X}$ and the momentum $\boldsymbol{P}$ of the particle described by a normalized quantum state, $|\Phi\rangle$, the associated information entropies are given by

$$S_N[\Phi; \boldsymbol{X}] = -\int \mathrm{d}^N \boldsymbol{x}\, |\Phi(\boldsymbol{x})|^2 \ln |\Phi(\boldsymbol{x})|^2 \quad (13)$$

$$S_N[\Phi; \boldsymbol{P}] = -\int \mathrm{d}^N \boldsymbol{p}\, \left|\tilde{\Phi}(\boldsymbol{p})\right|^2 \ln \left|\tilde{\Phi}(\boldsymbol{p})\right|^2 \quad (14)$$

respectively, where $\Phi(\boldsymbol{x}) = \langle \boldsymbol{x}|\Phi\rangle$ and $\tilde{\Phi}(\boldsymbol{p}) = \langle \boldsymbol{p}|\Phi\rangle$. Then, the information entropic uncertainty relation proved in [9] states that the sum of these quantities has the following irreducible lower bound:

$$U_N[\Phi; \boldsymbol{X}, \boldsymbol{P}] \equiv S_N[\Phi; \boldsymbol{X}] + S_N[\Phi; \boldsymbol{P}] \geqslant N(1 + \ln\pi). \tag{15}$$

This formulation of the uncertainty principle is to be compared with the ordinary Heisenberg-type formulation

$$\sqrt{(\Delta_\Phi \boldsymbol{X})^2 (\Delta_\Phi \boldsymbol{P})^2} \geqslant \frac{N}{2} \tag{16}$$

where $(\Delta_\Phi \boldsymbol{Q})^2 \equiv \langle\Phi|\boldsymbol{Q}^2|\Phi\rangle - \langle\Phi|\boldsymbol{Q}|\Phi\rangle^2$. In the present case of $\Phi(\boldsymbol{x}) = \phi(\boldsymbol{x})$ in equation (11), $\left(\Delta_\phi \boldsymbol{X}\right)^2$ is divergent, and therefore no information can be obtained on $\left(\Delta_\phi \boldsymbol{P}\right)^2$ from equation (16). In this context, it has recently been discussed in [10] that, in such a situation, the information theoretic approach is more useful than the Heisenberg-type formulation. In marked contrast with equation (16), both $S_N[\Phi; \boldsymbol{X}]$ and $S_N[\Phi; \boldsymbol{P}]$ tend to be finite for a wide class of normalizable states $\{|\Phi\rangle\}$. In fact, for the CL wavepacket $\phi(\boldsymbol{x})$, $S_N[\phi; \boldsymbol{X}]$ is calculated to be

$$S_N[\phi; \boldsymbol{X}] = \frac{N+1}{2}\psi\left(\frac{N+1}{2}\right) - \ln\Gamma\left(\frac{N+1}{2}\right) + \frac{N+1}{2}(\ln 4\pi + \gamma). \tag{17}$$

On the other hand, $S_N[\phi; \boldsymbol{P}]$ is calculated using equation (12) as follows:

$$S_N[\phi; \boldsymbol{P}] = \ln\left\{\frac{2^{(N-3)/2}\pi^{(N+1)/2}\left[\Gamma\left(\frac{N+1}{4}\right)\right]^2}{\Gamma\left(\frac{N+1}{2}\right)}\right\} - \frac{\Gamma\left(\frac{N+1}{2}\right)}{2^{(N-5)/2}\sqrt{\pi}\,\Gamma\left(\frac{N}{2}\right)\left[\Gamma\left(\frac{N+1}{4}\right)\right]^2} \times I_N \tag{18}$$

where

$$I_N = \int_0^\infty \mathrm{d}p\, p^{(N-1)/2}\left|K_{(N-1)/4}(p)\right|^2 \ln\left[p^{-(N-1)/2}\left|K_{(N-1)/4}(p)\right|^2\right]. \tag{19}$$

Therefore, we obtain

$$U_N[\phi; \boldsymbol{X}, \boldsymbol{P}] = \frac{N+1}{2}\left[\psi\left(\frac{N+1}{2}\right) + \gamma\right] + \ln\left\{2^{(3N-1)/2}\pi^{N+1}\left[\frac{\Gamma\left(\frac{N+1}{4}\right)}{\Gamma\left(\frac{N+1}{2}\right)}\right]^2\right\} - \frac{\Gamma\left(\frac{N+1}{2}\right)}{2^{(N-5)/2}\sqrt{\pi}\,\Gamma\left(\frac{N}{2}\right)\left[\Gamma\left(\frac{N+1}{4}\right)\right]^2} \times I_N. \tag{20}$$

$S_N[\phi; \boldsymbol{X}]$ monotonically increases with respect to N. Using equations (8) and (9) for large N, we see that it asymptotically behaves as follows:

$$S_N[\phi; \boldsymbol{X}] \cong \frac{N}{2}(\ln 4\pi + \gamma + 1). \tag{21}$$

On the other hand, it does not seem to be possible to evaluate $S_N[\phi; \boldsymbol{P}]$ analytically. Therefore, we have performed its numerical evaluation. We have found that, like $S_N[\phi; \boldsymbol{X}]$, $S_N[\phi; \boldsymbol{P}]$ also asymptotically scales linearly with respect to N for large values of N.

In figure 1, we present the plot of $U_N[\phi; \boldsymbol{X}, \boldsymbol{P}]/N$ with respect to N. We observe that this quantity monotonically approaches the rigorous lower bound $1 + \ln\pi$. This may be a result of interest, since so far it has been believed in the literature that the lower bound can be reached only by a class of Gaussian wavepackets.

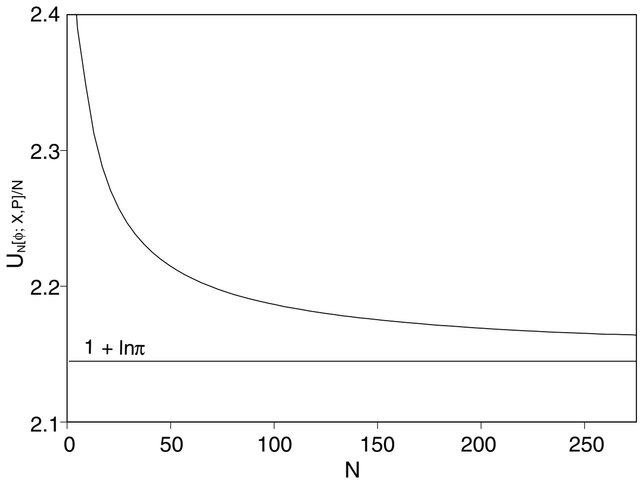

Figure 1. Plot of $U_N\,[\phi;\, \boldsymbol{X}, \boldsymbol{P}]\,/N$ with respect to N in arbitrary units.

5. Conclusion

We have studied the statistical properties of the multivariate CL distribution based on the information theoretic approach. We have discussed correlation measured by the mutual entropy associated with this distribution and analytically found its simple asymptotic linear scaling behaviour with respect to the dimensionality of the distribution. We have also discussed the information entropic uncertainty in the quantum CL wavepacket. We have shown how the position and momentum uncertainties also scale linearly with respect to the dimensionality asymptotically and their sum tends to the lower bound.

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